In 1766, he returned to Russia at the invitation of Catherine the Great. His eyesight had deteriorated over the years, and soon after his return to Russia he became totally blind. Incredibly, his blindness made little impact on his mathematical output, for he wrote several books and over 400 papers while blind. He remained busy and active until the day of his death.

Euler's productivity was remarkable: he wrote textbooks on physics, algebra, calculus, real and complex analysis, analytic and differential geometry, and the calculus of variations. He also wrote hundreds of original papers, many of which won prizes. A current edition of his collected works consists of 74 volumes.

depend on $x_1 < x_2$ or $x_1 > x_2$

Exercises for Section 3.3

1. Let $x_1 := 8$ and $x_{n+1} := \frac{1}{2}x_n + 2$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone. Find the limit.
2. Let $x_1 > 1$ and $x_{n+1} := 2 - 1/x_n$ for $n \in \mathbb{N}$. Show that (x_n) is bounded and monotone. Find the limit.
3. Let $x_1 \geq 2$ and $x_{n+1} := 1 + \sqrt{x_n - 1}$ for $n \in \mathbb{N}$. Show that (x_n) is decreasing and bounded below by 2. Find the limit.
4. Let $x_1 := 1$ and $x_{n+1} := \sqrt{2 + x_n}$ for $n \in \mathbb{N}$. Show that (x_n) converges and find the limit.
5. Let $y_1 := \sqrt{p}$, where $p > 0$, and $y_{n+1} := \sqrt{p + y_n}$ for $n \in \mathbb{N}$. Show that (y_n) converges and find the limit. [*Hint:* One upper bound is $1 + 2\sqrt{p}$.]
   $1+\sqrt{p}$ is an upper bd.
6. Let $a > 0$ and let $z_1 > 0$. Define $z_{n+1} := \sqrt{a + z_n}$ for $n \in \mathbb{N}$. Show that (z_n) converges and find the limit. $\xi = \max\{1, z_1\}$. Then $z_n \leq \xi + \sqrt{a}$ $\forall n$.
7. Let $x_1 := a > 0$ and $x_{n+1} := x_n + 1/x_n$ for $n \in \mathbb{N}$. Determine if (x_n) converges or diverges.
   increasing and must be unbounded (otherwise $\ell = \ell + 1/\ell$)
8. Let (a_n) be an increasing sequence, (b_n) a decreasing sequence, and assume that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Show that $\lim(a_n) \leq \lim(b_n)$, and thereby deduce the Nested Intervals Property 2.5.2 from the Monotone Convergence Theorem 3.3.2.
   $u \in A$ or $u \notin A$
9. Let A be an infinite subset of $\mathbb{R}$ that is bounded above and let $u := \sup A$. Show there exists an increasing sequence (x_n) with $x_n \in A$ for all $n \in \mathbb{N}$ such that $u = \lim(x_n)$.
10. Let (x_n) be a bounded sequence, and for each $n \in \mathbb{N}$ let $s_n := \sup\{x_k : k \geq n\}$ and $t_n := \inf\{x_k : k \geq n\}$. Prove that (s_n) and (t_n) are monotone and convergent. Also prove that if $\lim(s_n) = \lim(t_n)$, then (x_n) is convergent. [One calls $\lim(s_n)$ the **limit superior** of (x_n), and $\lim(t_n)$ the **limit inferior** of (x_n).]

$5, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$

17. Use a calculator to compute e_n for $n = 50$, $n = 100$, and $n = 1{,}000$.

Section 3.4 Subsequences and the Bolzano-Weierstra

In this section we will introduce the notion of a subsequence of a seque Informally, a subsequence of a sequence is a selection of terms from such that the selected terms form a new sequence. Usually the selection purpose. For example, subsequences are often useful in establishing th divergence of the sequence. We will also prove the important existenc the Bolzano-Weierstrass Theorem, which will be used to establish a n results.

3.4.1 Definition Let $X = (x_n)$ be a sequence of real numbers and $n_k < \cdots$ be a strictly increasing sequence of natural numbers. Then the given by

$$(x_{n_1}, x_{n_2}, \cdots, x_{n_k}, \cdots)$$

is called a **subsequence** of X.

For example, if $X := (\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots)$, then the selection of even inc the subsequence

$$X' = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \cdots, \frac{1}{2k}, \cdots\right),$$

where $n_1 = 2, n_2 = 4, \cdots, n_k = 2k, \cdots$. Other subsequences of X lowing:

$$\left(\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \cdots, \frac{1}{2k-1}, \cdots\right), \quad \left(\frac{1}{2!}, \frac{1}{4!}, \frac{1}{6!}, \cdots, \frac{1}{(2k)}\right.$$

The following sequences are *not* subsequences of $X = (1/n)$:

$$\left(\frac{1}{2}, \frac{1}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \cdots\right), \quad \left(\frac{1}{1}, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \cdots\right.$$